2. LINEAR TRANSFORMATIONS

§2.1. Linear Transformations

Let U, V be vector spaces over the field F. A function (map) f:U→V is called a **linear transformation** if:

- $f(u_1 + u_2) = f(u_1) + f(u_2)$ for all $u_1, u_2 \in U$ and
- $f(\lambda u) = \lambda f(u)$ for all $u \in U$ and $\lambda \in F$.

Example 1:

(1) Let F^n be the space of all n-dimensional column vectors over F and let A be any $n \times n$ matrix over F. Then $f(v) = Av$ is a linear transformation.

(2) Let E be the 3-dimensional Euclidean space and let OP be a line through the origin.

Let R be a rotation about OP through a certain angle.

Then R is a linear transformation.

This because a rotation will take a parallelogram to a congruent parallelogram and so R preserves vector addition. Also, a vector gets rotated to one of the same length, and so R satisfies the second property of the definition.

(3) Let Diff($\mathbb{R}$) be the space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Then differentiation is a linear transformation.

That is $\dfrac{d(y_1 + y_2)}{dx} = \dfrac{dy_1}{dx} + \dfrac{dy_2}{dx}$ and $\dfrac{d(\lambda y)}{dx} = \lambda\dfrac{dy}{dx}$.

(4) Let $\mathbb{R}^2$ be the space of real vectors $\mathbf{v} = (x, y)$ and let $\mathbf{a} \in \mathbb{R}^2$.

Then $f(\mathbf{v}) = \mathbf{a}.\mathbf{v}$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$.

Example 2: In the days before Sodoku they had things called magic squares. These were square tables (we would call them matrices) with a number in each square. Every row, every column, and both diagonals had to have the same total. It was usually assumed that the entries are all positive integers and that there are no repetitions, but these aren't strictly necessary. Albrecht Dürer, in his engraving Melancholia, gives an example of a 4×4 magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

A most uninteresting 4 × 4 magic square is:

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

and an even more boring example is the zero matrix.

Let us denote the set of $n \times n$ magic squares, with real number entries, as Q_n. This is a vector space under matrix addition and scalar multiplication. The function $T:Q_n \to \mathbb{R}$, where T(M) is the common total for the magic square M, is a linear transformation.

Theorem 1: If $f:U \to V$ is a linear transformation:
  (1) $f(-v) = -f(v)$ for all $v \in V$;
  (2) $f(0) = 0$,
Proof: (1) $f(-v) = f((-1)v) = -f(v)$.
  (2) $f(\mathbf{0}) = f(0.\mathbf{0}) = 0f(\mathbf{0}) = \mathbf{0}$. ✋☺

NOTE: In (2), for clarity, we distinguish between the zero scalar and the zero vector by making the zero vector bold. However keep in mind that they could well be the same thing in certain examples.

Recall that the **product** of two functions f, g is fg where $(fg)(x) = g(f(x))$. We first apply the left-hand map and then the right-hand one. Don't confuse this with

composition of functions, which is the same idea backwards. The **composition** of f, g is f ∘ g where

$$(f \circ g)(x) = f(g(x)).$$

Theorem 2: The product of two linear transformations is a linear transformation.

Proof: Suppose f:U → V and g:V → W are linear transformations.

Then for all $u_1, u_2 \in U$,

$(fg)(u_1 + u_2) = g(f(u_1 + u_2))$ by definition of fg

$\qquad\qquad = g(f(u_1) + f(u_2))$ since f is linear

$\qquad\qquad = g(f(u_1)) + g(f(u_2))$ since g is linear

$\qquad\qquad = (fg)(u_1) + (fg)(u_2)$ by definition of fg.

The other property for linearity is proved similarly, so fg:U → W is linear. ✋☺

§2.2. Kernels and Images

The **image** of a linear transformation f:U → V is defined as for any function: **im(f)** = {f(u) | u ∈ U}.

The **kernel** of f is defined to be: **ker(f)** = {u ∈ U | f(u) = 0}.

Theorem 3: Suppose U, V are vector spaces over F.

If f:U $\rightarrow$ V is a linear transformation then:

(1) im(f) is a subspace of V;

(2) ker(f) is a subspace of U and

(3) dim im(f) + dim ker(f) = dim(U).

Proof:

(1) Let f(u$_1$), f(u$_2$) $\in$ im(f).

Then f(u$_1$) + f(u$_2$) = f(u$_1$ + u$_2$) $\in$ im(f) and so im(f) is closed under addition.

Let f(u) $\in$ im(f) and let $\lambda \in$ F. Then λf(u) = f(λu) $\in$ im(f) and so im(f) is closed under scalar multiplication.

Hence im(f) is a subspace of V.

(2) Let u$_1$, u$_2$, u $\in$ ker(f). Then f(u$_1$) = f(u$_2$) = f(u) = 0.

Now f(u$_1$ + u$_2$) = f(u$_1$) + f(u$_2$) = 0 + 0 = 0 and so ker(f) is closed under addition.
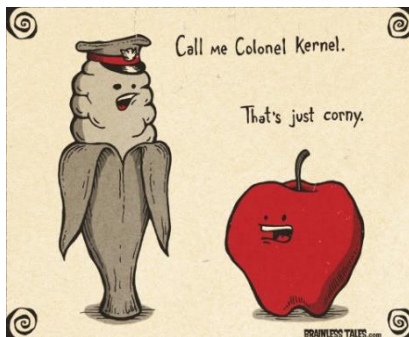
Also f(λu) = λf(u) = λ0 = 0 and so ker(f) is closed under scalar multiplication.

Hence ker(f) is a subspace of V.

(3) Let u$_1$, u$_2$, …, u$_k$ be a basis for ker(f).

Extend this to a basis u$_1$, u$_2$, …, u$_n$ for U.

Then we'll show that f(u$_{r+1}$), f(u$_{r+2}$), …, f(u$_n$) is a basis for im(f).

(a) $f(u_{r+1})$, $f(u_{r+2})$, …, $f(u_n)$ span im(f):

Let $f(u) \in$ im(f), where $u \in U$.

Then $u = \lambda_1 u_1 + \lambda_2 u_2 + \ldots \lambda_k u_k + \lambda_{k+1} u_{k+1} + \ldots + \lambda_n u_n$ for some λ_i's $\in F$. Hence

$f(u) = f(\lambda_1 u_1 + \lambda_2 u_2 + \ldots \lambda_k u_k + \lambda_{k+1} u_{k+1} + \ldots + \lambda_n u_n)$

$\quad = \lambda_1 f(u_1) + \lambda_2 f(u_2) + \ldots \lambda_k f(u_k)$

$\qquad\qquad\qquad\qquad\qquad + \lambda_{k+1} f(u_{k+1}) + \ldots + \lambda_n f(u_n)$

$\quad = 0 + 0 + \ldots + 0 + \lambda_{k+1} f(u_{k+1}) + \ldots + \lambda_n f(u_n)$

$\quad = \lambda_{k+1} f(u_{k+1}) + \ldots + \lambda_n f(u_n)$.

(b) $f(u_{r+1})$, $f(u_{r+2})$, …, $f(u_n)$ are linearly independent:

Suppose $\lambda_{k+1} f(u_{k+1}) + \ldots + \lambda_n f(u_n) = 0$.

Then $f(\lambda_{k+1} u_{k+1} + \ldots + \lambda_n u_n) = 0$.

Hence $\lambda_{k+1} u_{k+1} + \ldots + \lambda_n u_n \in$ ker(f).

So $\lambda_{k+1} u_{k+1} + \ldots + \lambda_n u_n = y_1 u_1 + y_2 u_2 + \ldots + y_k u_k$ for some y_i's $\in F$.

Hence $y_1 u_1 + y_2 u_2 + \ldots + y_k u_k - \lambda_{k+1} u_{k+1} - \ldots - \lambda_n u_n = 0$ and since these vectors are linearly independent:

$y_1 = y_2 = \ldots = y_k = x_{k+1} = \ldots = x_n = 0$. ✋☺

      We define the **rank** of a linear transformation f:U $\to$ V to be dim im(f) and its **nullity** is defined to be dim ker(f). The above theorem can be expressed by saying:

rank + nullity = dimension of the space you're mapping from.

Example 3: Projecting a point in 3-dimensional Euclidean space onto the *x-y* plane is a linear transformation. Its image is the *x-y* plane, of dimension 2. Its kernel is the z-axis, of dimension 1. The above theorem is verified by the fact that $2 + 1 = 3$.

Example 4: What is the dimension of Q_3, the space of all 3×3 real magic squares?
Solution: The kernel of the 'total' linear transformation T given in Example 2 is the set of those whose total is zero. It is easy to give a general description of the elements of ker(T).

Let's begin by ensuring that the rows and columns each total zero.

a	b	$-a-b$
c	d	$-c-d$
$-a-c$	$-b-d$	$a+b+c+d$

For the diagonals to total zero we need to satisfy the system of equations:
$$\begin{cases} 2a + b + c + 2d = 0 \\ -2a - b - c + d = 0 \end{cases}$$

Adding the equations gives $3d = 0$, so $d = 0$.
Then $c = -2a - b$. This gives:

a	b	$-a-b$
$-2a-b$	0	$2a+b$
$a+b$	$-b$	$-a$

By taking $a = 1$, $b = 0$ and $b = 1$, $a = 0$ we can obtain a basis for ker (T):

1	0	−1
−2	0	2
1	0	−1

0	1	−1
−1	0	1
1	−1	0

So T has nullity 2. The image of T is clearly $\mathbb{R}$, which is 1-dimensional, so dim $Q_3 = 1 + 2 = 3$.

An **isomorphism** f:U $\rightarrow$ V (often called a **non-singular linear transformation** or an **invertible linear transformation**) is a linear transformation that is both 1-1 and onto. Like any 1-1 and onto function it has an inverse f^{-1}. In this case the image is V and the kernel is {0}. So the rank is dim(V) and the nullity is 0. Hence dim(V) + 0 = dim(U).

If there exists a isomorphism f:U $\rightarrow$ V we say that U and V are **isomorphic**, and write $U \cong V$. Since rank plus nullity is the dimension of the space you're mapping from, dim(V) = dim(U). Hence isomorphic vector spaces have the same dimension.

Conversely if U and V have the same dimension (over the same field) then they are isomorphic. One only has to take a basis for each, $\{u_1, ..., u_n\}$ and $\{v_1, ..., v_n\}$ and map $\lambda_1 u_1 + ... + \lambda_n u_n$ to $\lambda_1 v_1 + ... + \lambda_n v_n$.

Isomorphic objects in any algebraic setting can be thought of as being the same, when viewed in isolation.

Their structure is the same and they may only differ in notation. The isomorphism gives the coding that takes one type of notation to another. So we can paraphrase the above result by saying that over a given field there is only one vector space (up to isomorphism) of each dimension.

If f is an isomorphism from a vector space to itself we call it an **automorphism**. The most obvious automorphism is the identity map I, that takes each vector to itself.

Theorem 4: The inverse of an isomorphism is an isomorphism.
Proof: Let $f:U \to V$ be an isomorphism. Since f is 1-1 and onto, f^{-1} exists and $f f^{-1} = f^{-1}f = I$.
Let $v_1, v_2 \in V$.
Then $v_1 = f(u_1)$ and $v_2 = f(u_2)$ for some $u_1, u_2 \in U$.
So $f^{-1}(v_1 + v_2) = f^{-1}(f(u_1) + f(u_2))$
$$= f^{-1}(f(u_1 + u_2)), \text{ since f is linear}$$
$$= ff^{-1}(u_1 + u_2) = u_1 + u_2$$
$$= f^{-1}(v_1) + f^{-1}(v_2). ☝☺$$

The other property of linearity is proved similarly.

§2.3. Coordinates

With our definition of vector space there is no mention of the coordinates of a vector. Coordinates can only be defined once we fix on a particular basis.

If v_1, v_2, ... , v_n is a basis for the finite dimensional vector space V then every vector can be expressed as a linear combination

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n,$$

because the v_i span V,

and moreover the λ_i's are unique, for a given v, because the v_i are linearly independent.

We define the **coordinates** of such a v, relative the basis v_1, v_2, ... , v_n as $(\lambda_1, \lambda_2, \ldots , \lambda_n)$. If we have a different basis, the coordinates will be different.

Example 5: The coordinates of (3, 2, 5) relative to the standard basis are (3, 1, 5) because
$$(3, 2, 5) = 3(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1).$$
What are the coordinates relative to the basis
$$(1, 1, 0), (0, 2, 1), (1, 1, 1)?$$
Solution: We must write (3, 2, 5) in the form:
$$x(1, 1, 0) + y(0, 2, 1) + z(1, 1, 1).$$
This gives us a system of equations: $\begin{cases} x + z = 3 \\ x + 2y + z = 2 \\ y + z = 5 \end{cases}$.

Writing this as an augmented matrix and putting this into

echelon form we get (leaving out the vertical bar that represents the equal signs):

$$\begin{pmatrix} 1\ 0\ 1\ 3 \\ 2\ 1\ 1\ 2 \\ 0\ 1\ 1\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1\ 0\ 1\ \ \ 3 \\ 0\ 1\text{--}1\ \text{--}4 \\ 0\ 1\ 1\ \ \ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1\ 0\ 1\ \ \ 3 \\ 0\ 1\text{--}1\ \text{--}4 \\ 0\ 0\ 2\ \ \ 9 \end{pmatrix}.$$

So $z = 9/2$, $y = -4 + 9/2 = 1/2$ and $x = 3 - 9/2 = -3/2$.
So the required coordinate vector is $(-3/2, 1/2, 9/2)$.

Example 6: The functions $\sin x$ and $\cos x$ form a basis for the space of solutions to the differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

What is the coordinate vector of $\sin(x + \pi/6)$ relative to this basis?

Solution: $\sin(x + \pi/6) = \sin x \cos(\pi/6) + \cos x \sin(\pi/6)$
$$= \tfrac{1}{2}\sqrt{3}\ \sin x + \tfrac{1}{2} \cos x,$$

so the coordinate vector of $\sin(x + \pi/6)$ relative to the above basis is $(\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2})$.

§2.4. Change of Basis

Sometimes we'd like to change bases. We may start with one basis and then realise that another basis would be more convenient. Can we readily move from one coordinate vector to another, for the new basis?

A basis is not just a set of vectors, because the order in which we choose to write them is important. So a basis is more like a vector, except that the components are not

scalars, they are vectors. But we can write the basis as if it is a vector. We write $\alpha_1, \alpha_2, \ldots, \alpha_n$ as

$$[\alpha] = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

We'll write the coordinates of v relative to this basis as $(\alpha_1, \alpha_2, \alpha_3)$ where

$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; \alpha_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \; \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Putting this all together we could write the basis as:

$$[\alpha] = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$ which could be written more

simply as the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

We shall write the column vector giving the coordinates of v relative the basis $[\alpha]$ as $\left[\dfrac{v}{\alpha} \right]$.

Example 7: If $[\alpha]$ is the first basis in example 18 and $[\beta]$ is the second basis, and $v = (3, 2, 5)$, find $\left[\dfrac{v}{\alpha} \right]$ and $\left[\dfrac{v}{\beta} \right]$.

Solution: $\left[\dfrac{v}{\alpha} \right] = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ and $\left[\dfrac{v}{\beta} \right] = \begin{pmatrix} -3/2 \\ 1/2 \\ 9/2 \end{pmatrix}$.

It would be nice if we could easily move from one coordinate vector to the other, especially if we have several vectors to convert. We do this using the **change of basis matrix**. This has, as its columns, the coordinate vectors for each of the vectors in the first basis, relative to the second.

Let $[\alpha] = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $[\beta] = (\beta_1, \beta_2, \ldots, \beta_n)$ be two bases for the n-dimensional vector space V. We define $\left[\dfrac{\alpha}{\beta}\right] = \left(\left[\dfrac{\alpha_1}{\beta}\right] \left[\dfrac{\alpha_2}{\beta}\right] \left[\dfrac{\alpha_3}{\beta}\right]\right)$.

It will be an $n \times n$ matrix.

Example 8: Let $\alpha = ((7, 16), (2, -1))$, $\beta = ((2, 5), (1, 1))$. What is $\left[\dfrac{\alpha}{\beta}\right]$?

Solution: $(7, 16) = a(2, 5) + b(1, 1)$ for some a, b. Equating and solving we find that $a = 3$, $b = 1$.
These are the coordinates of $(7, 16)$ relative to β.
Similarly the coordinates of $(2, -1)$ are $(-1, 4)$.
So $\left[\dfrac{\alpha}{\beta}\right] = \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}$

Theorem 5: $[\alpha] = [\beta]\left[\dfrac{\alpha}{\beta}\right]$.

Proof:
Let the coordinate vector of α_j relative to the basis $[\beta]$ be
$$(a_{1j}, a_{2j}, \ldots, a_{nj}).$$
Then for each j, $\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \ldots + a_{nj}\beta_n$.

So $[\beta]\begin{bmatrix}\alpha\\\beta\end{bmatrix} = (\beta_1, \beta_2, \ldots, \beta_n)\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$

$$= (\alpha_1, \alpha_2, \ldots, \alpha_n). ✌☺$$

Example 9: Let $U = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$.
Let α be the standard bases for U and let
$$\beta = \{(1, 2), (1, -1)\}$$
be another basis.

$\alpha_1 = (1, 0) = (1/3)(1, 2) + (2/3)(1, -1)\ (1/3)\beta_1 + (2/3)\beta_2$.
$\alpha_2 = (0, 1) = (1/3)(1, 2) - (1/3)(1, -1) = (1/3)\beta_1 - (1/3)\beta_2$.

So $\begin{bmatrix}\alpha_1\\\beta\end{bmatrix} = \begin{pmatrix}1/3\\2/3\end{pmatrix}$ and $\begin{bmatrix}\alpha_2\\\beta\end{bmatrix} = \begin{pmatrix}1/3\\-1/3\end{pmatrix}$ and hence

$$\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{pmatrix}1/3 & 1/3\\2/3 & -1/3\end{pmatrix}.$$

$[\beta]\begin{bmatrix}\alpha\\\beta\end{bmatrix} = (\beta_1, \beta_2)\begin{pmatrix}1/3 & 1/3\\2/3 & -1/3\end{pmatrix}$

$\quad = ((1/3)\beta_1 + (2/3)\beta_2, (1/3)\beta_1 - (1/3)\beta_2)$

$\quad = (\alpha_1, \alpha_2)$

$\quad = [\alpha].$

§2.5. The Matrix of a Linear Transformation

Suppose U, V are vector spaces over the field F, of dimensions m, n respectively.

Let $f: U \rightarrow V$ be a linear transformation and let α be a basis for U and β a basis for V. We define the **matrix of f** relative to these bases to be:

$$\left[\frac{f(\alpha)}{\beta}\right] = \left(\left[\frac{f(\alpha_1)}{\beta}\right] \left[\frac{f(\alpha_2)}{\beta}\right] \quad \left[\frac{f(\alpha_m)}{\beta}\right]\right)$$

Example 10: Let $U = \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$,
$$V = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$
Let $f(x, y, z) = (x + y + z, 2x - y)$.
Let α be the standard bases of U and
let $\beta = \{(1, 2), (1, -1)\}$ (another basis for V).
$\alpha_1 = (1, 0, 0)$ and so $f(\alpha_1) = (1, 2)$.
$\alpha_2 = (0, 1, 0)$ and so $f(\alpha_2) = (1, -1)$.
$\alpha_3 = (0, 0, 1)$ and so $f(\alpha_3) = (1, 0)$.

$$\left[\frac{f(\alpha_1)}{\beta}\right] = \binom{1}{0}$$

$$\left[\frac{f(\alpha_1)}{\beta}\right] = \binom{0}{1}$$

$$\left[\frac{f(\alpha_1)}{\beta}\right] = \binom{1/3}{2/3} \text{ since } (1, 0) = (1/3)(1, 2) + (2/3)(1, -1).$$

So $\left[\dfrac{f(\alpha)}{\beta}\right] = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{pmatrix}$.

Theorem 6: $\left[\dfrac{f(v)}{\beta}\right] = \left[\dfrac{f(\alpha)}{\beta}\right]\left[\dfrac{v}{\alpha}\right].$

Proof: Suppose dim $U = m$, dim $V = n$.

Let $v = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \ldots + \lambda_m\alpha_m$, so that $\left[\dfrac{v}{\alpha}\right] = \begin{pmatrix} \lambda_1 \\ \ldots \\ \lambda_m \end{pmatrix}.$

Then $f(v) = \lambda_1 f(\alpha_1) + \lambda_2 f(\alpha_2) + \ldots + \lambda_m f(\alpha_m)$.

Suppose that $f(\alpha_i) = a_{1i}\beta_1 + a_{2i}\beta_2 + \ldots + a_{mi}\beta_m$ for each i.

Then $\left[\dfrac{f(\alpha)}{\beta}\right] = (a_{ij})$.

So we may express $f(v)$ as

$\lambda_1(a_{11}\beta_1 + a_{21}\beta_2 + \ldots + a_{mi}\beta_m) + \ldots$

$\ldots + \lambda_m(a_{1m}\beta_1 + a_{2m}\beta_2 + \ldots + a_{mm}\beta_m)$

$= (a_{11}\lambda_1 + \ldots + a_{1m}\lambda_m)\beta_1 + \ldots + (a_{m1}\lambda_1 + \ldots + a_{mn}\lambda_m)\beta_m.$

Hence $\left[\dfrac{f(v)}{\beta}\right] = \begin{pmatrix} a_{11}\lambda_1 + \ldots + a_{1m}\lambda_m \\ \ldots \\ a_{m1}\lambda_1 + \ldots + a_{mn}\lambda_m \end{pmatrix} = \left[\dfrac{f(\alpha)}{\beta}\right]\left[\dfrac{v}{\alpha}\right].$ ✋☺

      This means that, relative to these bases, the action of the linear transformation f is equivalent to multiplying the coordinate vector, on the left, by $\left[\dfrac{f(\alpha)}{\beta}\right]$. If you have difficulty remembering which symbol goes where in the formula $\left[\dfrac{f(v)}{\beta}\right] = \left[\dfrac{f(\alpha)}{\beta}\right]\left[\dfrac{v}{\alpha}\right]$ just remove the parentheses

and brackets and write $\dfrac{fv}{\beta} = \dfrac{f\alpha}{\beta} \cdot \dfrac{v}{\alpha}$. Then it is just as if you are cancelling the α's.

Example 11: Let $v = (1, 1, 1)$ in example 9.

Then $\left[\dfrac{v}{\alpha}\right] = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$f(v) = (3, 1) = (4/3)(1, 2) + (5/3)(1, -1) = (4/3)\beta_1 + (5/3)\beta_2$.

So $\left[\dfrac{f(v)}{\beta}\right] = \begin{pmatrix} 4/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left[\dfrac{f(\alpha)}{\beta}\right]\left[\dfrac{v}{\alpha}\right]$.

Theorem 7: Suppose U, V, W are vector spaces over F, with bases α, β, γ respectively.
Suppose $f:U \to V$ and $g:V \to W$ are linear transformations.

$$\text{Then } \left[\dfrac{(fg)(\alpha)}{\gamma}\right] = \left[\dfrac{g(\beta)}{\gamma}\right]\left[\dfrac{f(\alpha)}{\beta}\right].$$

Proof: This because fg is achieved by first carrying out f and following it by g. In terms of the coordinate vectors this is equivalent to multiplying $\left[\dfrac{v}{\alpha}\right]$ on the left by the matrix for f and then multiplying on the left by the matrix for g. This is why the order of the matrices is the reverse of the linear transformations. If you remember this you can remove the parentheses and cancel the β's on the RHS as a way for checking that you have remembered the formula. ✋☺

Example 12: Let $U = \mathbb{R}^3$, $V = \mathbb{R}^2$ and $W = \mathbb{R}^4$.

Let $\alpha = ((1, 1, 1), (1, 1, 0), (0, 1, 1))$ be the basis for U,

   $\beta = ((2, 3), (-1, 0))$ be the basis for V and

   $\gamma = ((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (1, 2, 3, 4))$ be the basis for W.

Let $f: U \to V$ be defined by $f((x, y, z)) = (x + y, y + z)$ and let $g: V \to W$ be defined by $g(x, y) = (x, x + y, x - y, -y)$. Then $f(\alpha_1) = (2, 2)$, $f(\alpha_2) = (2, 1)$ and $f(\alpha_3) = (1, 2)$, so

$$\left[\frac{f(\alpha)}{\beta}\right] = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & -4/3 & 1/3 \end{pmatrix}.$$

Also $g(\beta_1) = (2, 5, -1, -3)$ and $g(\beta_2) = (-1, -1, -1, 0)$ so

$$\left[\frac{g(\beta)}{\gamma}\right] = \begin{pmatrix} 4 & -2 \\ 9 & -3 \\ 5 & -4 \\ -2 & 1 \end{pmatrix}.$$

$$\left[\frac{g(\beta)}{\gamma}\right]\left[\frac{f(\alpha)}{\beta}\right] = \begin{pmatrix} 4 & -2 \\ 9 & -3 \\ 5 & -4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & -4/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 2 \\ 8 & 7 & 5 \\ 6 & 7 & 2 \\ -2 & -2 & -1 \end{pmatrix}.$$

$(fg)(\alpha_1) = (2, 4, 0, -2)$, $(fg)(\alpha_2) = (2, 3, 1, -1)$ and $(fg)(\alpha_3) = (1, 3, -1, -2)$

$$\left[\frac{(fg)(\alpha_1)}{\gamma} \right] = \begin{pmatrix} 4 & 4 & 2 \\ 8 & 7 & 5 \\ 6 & 7 & 2 \\ -2 & -2 & -1 \end{pmatrix}.$$

The matrix of a linear transformation $f: U \to V$ depends on the bases for U and V that we choose. But with different bases these matrices would be related to one another.

Theorem 8: Suppose α, β are two bases for U and suppose that γ, δ are two bases for V and that $f: U \to V$ is a linear transformation.

$$\text{Then} \left[\frac{f(\beta)}{\delta} \right] = \left[\frac{\gamma}{\delta} \right] \left[\frac{f(\alpha)}{\gamma} \right] \left[\frac{\beta}{\alpha} \right].$$

Proof: Let $\beta_i = a_{1i}\alpha_1 + \ldots + a_{mi}\alpha_m$ and
$\qquad\qquad \delta_i = b_{1i}\alpha_1 + \ldots + b_{mi}\alpha_m$ for each i.

Let $A = \left[\frac{\beta}{\alpha} \right] = (a_{ij})$ and $B = \left[\frac{\delta}{\gamma} \right] = (b_{ij})$.

Let $f(\alpha_i) = c_{1i}\gamma_1 + \ldots + c_{ni}\gamma_n$ for each i.

Let $C = \left[\frac{f(\alpha)}{\gamma} \right] = (c_{ij})$.

Now $f(\beta_i) = a_{1i}f(\alpha_1) + \ldots + a_{mi}f(\alpha_m)$

$\qquad\qquad = a_{1i}(c_{11}\gamma_1 + \ldots + c_{n1}\gamma_n) + \ldots$

$\qquad\qquad\qquad\qquad \ldots + a_{mi}(c_{1m}\gamma_1 + \ldots + c_{nm}\gamma_n)$

$\qquad\qquad = (a_{1i}c_{11} + \ldots a_{m1}c_{1m})\gamma_1 + \ldots$

$\qquad\qquad\qquad\qquad \ldots + (a_{1i}c_{n1} + \ldots + a_{mi}c_{nm})\gamma_n$

So $\left[\dfrac{f(\beta_i)}{\gamma}\right] = \begin{pmatrix} a_{1i}c_{11} + \dots + a_{m1}c_{1m} \\ \dots \\ a_{1i}c_{n1} + \dots + a_{mi}c_{nm} \end{pmatrix}$

$= \begin{pmatrix} c_{11}a_{1i} + \dots + c_{1m}a_{mi} \\ \dots \\ c_{n1}c_{n1} + \dots + c_{nm}a_{mn} \end{pmatrix}.$

This is the i'th column of CA.

Now $\left[\dfrac{f(\beta_i)}{\delta}\right] = \left[\dfrac{\gamma}{\delta}\right]\left[\dfrac{f(\beta_i)}{\gamma}\right].$

This is the i'th column of DCA where $D = \left[\dfrac{\gamma}{\delta}\right].$

Hence $\left[\dfrac{f(\beta)}{\delta}\right] = DCA = \left[\dfrac{\gamma}{\delta}\right]\left[\dfrac{f(\alpha)}{\gamma}\right]\left[\dfrac{\beta}{\alpha}\right].$ 🖐☺

Again, if you have trouble remembering which symbols go in the numerator and which in the denominator, just ignore the brackets and parentheses and check that, if they were fractions, the right hand side would cancel to equal the left hand side: $\dfrac{f\beta}{\delta} = \dfrac{\gamma}{\delta} \cdot \dfrac{f\delta}{\gamma} \cdot \dfrac{\beta}{\alpha}.$

Example 13: Let $U = \mathbb{R}^2$ and $V = \mathbb{R}^3$.
Let α be the standard basis for U and let
$\beta = ((1, 1), (1, -1))$ be another basis.
Let $\gamma = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ be a basis for V and let δ be the standard basis for V.
Let $f: U \to V$ be the linear transformation
$$f(x, y) = (y, x + y, x).$$

Then $\begin{bmatrix}\gamma\\\delta\end{bmatrix} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$, $\begin{bmatrix}f(\alpha)\\\gamma\end{bmatrix} = \begin{pmatrix}0&1\\1&0\\0&0\end{pmatrix}$ and $\begin{bmatrix}\beta\\\alpha\end{bmatrix} = \begin{pmatrix}1&1\\1&-1\end{pmatrix}$.

Hence $\begin{bmatrix}\gamma\\\delta\end{bmatrix}\begin{bmatrix}f(\alpha)\\\gamma\end{bmatrix}\begin{bmatrix}\beta\\\alpha\end{bmatrix} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

$$= \begin{pmatrix}0&1\\1&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$= \begin{pmatrix}1&-1\\2&0\\1&1\end{pmatrix} = \begin{bmatrix}f(\beta)\\\delta\end{bmatrix}.$$

Theorem 9: If α, β are bases for a vector space V then:
$$\begin{bmatrix}\beta\\\alpha\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}^{-1}.$$

Proof: Taking f to be the identity linear transformation, and $\alpha = \gamma$, $\beta = \delta$ in theorem 7 we get
$$I = \begin{bmatrix}\beta\\\beta\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}\begin{bmatrix}\alpha\\\alpha\end{bmatrix}\begin{bmatrix}\beta\\\alpha\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}\begin{bmatrix}\beta\\\alpha\end{bmatrix}. \ ☝☺$$

Theorem 10: Suppose α, β are two bases for V and that f:V $\rightarrow$ V is a linear transformation.
$$\text{Then } \begin{bmatrix}f(\beta)\\\beta\end{bmatrix} = \begin{bmatrix}\beta\\\alpha\end{bmatrix}^{-1}\begin{bmatrix}f(\alpha)\\\alpha\end{bmatrix}\begin{bmatrix}\beta\\\alpha\end{bmatrix}.$$

Proof: The result follows from theorems 6 and 7. ☝☺

If we removed brackets and parentheses this would appear as: $\dfrac{f\beta}{\beta} = \dfrac{\alpha}{\beta} \cdot \dfrac{f\alpha}{\alpha} \cdot \dfrac{\beta}{\alpha}$.

We can summarise these last theorems as follows:

$\left[\dfrac{\alpha}{\alpha}\right] = I$	$\left[\dfrac{\beta}{\alpha}\right] = \left[\dfrac{\alpha}{\beta}\right]^{-1}$	$\left[\dfrac{v}{\beta}\right] = \left[\dfrac{\alpha}{\beta}\right]\left[\dfrac{v}{\alpha}\right]$	$[\alpha] = [\beta]\left[\dfrac{\alpha}{\beta}\right]$
$\left[\dfrac{F(v)}{\beta}\right] = \left[\dfrac{F(\alpha)}{\beta}\right]\left[\dfrac{v}{\alpha}\right]$		$\left[\dfrac{FG(\alpha)}{\gamma}\right] = \left[\dfrac{G(\beta)}{\gamma}\right]\left[\dfrac{F(\alpha)}{\beta}\right]$	
$\left[\dfrac{F(\beta)}{\delta}\right] = \left[\dfrac{\gamma}{\alpha}\right]\left[\dfrac{F(\alpha)}{\gamma}\right]\left[\dfrac{\beta}{\alpha}\right]$		$\left[\dfrac{F(\beta)}{\beta}\right] = \left[\dfrac{\beta}{\alpha}\right]^{-1}\left[\dfrac{F(\alpha)}{\alpha}\right]\left[\dfrac{\beta}{\alpha}\right]$	

EXERCISES FOR CHAPTER 2

Exercise 1: Let $\mathbb{R}[x]$ be the space of all real polynomials in x. Define $\Phi: \mathbb{R}[x] \to \mathbb{R}^3$ by $\Phi(f(x)) = (f(1), f(2), f(3))$. Show that Φ is a linear transformation and find ker Φ and im Φ.

Exercise 2: Find the coordinates of $(1, -5, 7)$ relative to the basis $(2, 1, -1)$, $(1, 0, 1)$, $(-1, 1, 5)$.

Exercise 3: A parallelogram in $\mathbb{R}^3$ has vertices $(1, 2, 3)$, $(6, 3, 5)$, $(-2, 6, 10)$, $(3, 7, 6)$. Find the coordinates of these points relative to the basis

$$(4, 1, 3), (-1, 2, 2), (8, -11, -7).$$

Exercise 4: Find the matrix of the linear transformation

$$T(x, y, z) = (x + 2y - 5z, 2x + 3y - z, 3y - 4z)$$

with respect to:
    (i) the standard basis, α;
    (ii) the basis $\beta = \{(1, 1, 0), (2, -1, 2), (3, 0, -4)\}$.

Exercise 5: Let $U = \langle xe^x, e^x \rangle$ and $V = \langle 1, x \rangle$.
Let D_U, D_V be the differentiation operator restricted to U, V respectively. Find the rank and nullity of each of these.

Exercise 6:
(i) Show that $\beta = \{\sin 2x, \sin^2 x, \cos^2 x, 2x\}$ is a basis for $V = \langle \sin 2x, \cos 2x, \sin^2 x, \cos^2 x, x \rangle$.
(ii) Show that differentiation is a linear transformation from V to V.
(iii) Find the matrix of differentiation relative to this basis.

SOLUTIONS FOR CHAPTER 2

Exercise 1:

$$\Phi[f(x) + g(x)] = (f(1) + g(1), f(2) + g(2), f(3) + g(3))$$
$$= (f(1), f(2), f(3)) + (g(1), g(2), g(3))$$
$$= \Phi[f(x)] + \Phi[g(x)].$$
$$\Phi[k(f(x))] = (kf(1), kf(2), kf(3))$$
$$= k(f(1), f(2), f(3)]$$
$$= k\Phi[f(x)].$$

ker Φ:

ker Φ = {f(x) | (f(1), f(2), f(3)) = (0, 0, 0)}

    = {f(x) | f(1) = 0, f(2) = 0, f(3) = 0}

    = $(x-1)(x-2)(x-3)\,\mathbb{R}[x]$

  (this is the set of all multiples of $(x-1)(x-2)(x-3)$

So ker Φ = $(x-1)(x-2)(x-3)\,\mathbb{R}[x]$.

im Φ: Suppose $(a, b, c) \in \mathbb{R}^3$.

We must find a polynomial f(x) such that

$$f(1) = a, \; f(2) = b \text{ and } f(3) = c.$$

Consider a polynomial $f(x) = px^2 + qx + r$.

If this satisfies the requirements then

$$\begin{cases} p + q + r = a \\ 4p + 2q + r = b \\ 9p + 3q + r = c \end{cases}$$

So we need to show that this system is consistent.

We'd would need to be able to solve the equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}$ is a Vandermonde matrix it is invertible

and so this equation has a unique solution. Thus there *does* exist a polynomial satisfying the requirements.

      We can explicitly find such a quadratic, without solving the above equations, very simply:

Take $f(x) = (x - 1)(x - 2)(c/3) + (x - 1)(x - 3)(-b)$
$$+ (x - 2)(x - 3)(a/2).$$
Clearly $f(1) = a$, $f(2) = b$, $f(3) = c$.

Exercise 2: We must write $(1, -5, 7)$ in the form:
$$x(2, 1, -1) + y(1, 0, 1) + z(3, 2, 5).$$
This gives us a system of equations that can be represented as:

$$\begin{pmatrix} 2 & 1 & -1 & | & 1 \\ 1 & 0 & 1 & | & -5 \\ -1 & 1 & 5 & | & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & -5 \\ 2 & 1 & -1 & | & 1 \\ -1 & 1 & 5 & | & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & -5 \\ 0 & 1 & -3 & | & 11 \\ 0 & 1 & 6 & | & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & | & -5 \\ 0 & 1 & -3 & | & 11 \\ 0 & 0 & 9 & | & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & -5 \\ 0 & 1 & -3 & | & 11 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}.$$

So $z = -1$, $y = 11 - 3 = 8$, $x = -5 + 1 = -4$.
Hence the coordinates of $(1, -5, 7)$ relative to the basis $(2, 1, -1)$, $(1, 0, 1)$, $(3, 2, 5)$ are $(-4, 8, -1)$.

Alternatively, we can write α = standard basis, β = this new basis and $v = (1, -5, 7)$.

Then $\left[\dfrac{v}{\alpha} \right] = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix}$ and $\left[\dfrac{\beta}{\alpha} \right] = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{pmatrix}$.

Now $\begin{bmatrix} v \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\begin{bmatrix} v \\ \alpha \end{bmatrix}$ and $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{pmatrix}^{-1}$

$$= -\frac{1}{9}\begin{pmatrix} -1 & -6 & 1 \\ -6 & 9 & -3 \\ 1 & -3 & -1 \end{pmatrix}.$$

Hence $\begin{bmatrix} v \\ \beta \end{bmatrix} = -\frac{1}{9}\begin{pmatrix} -1 & -6 & 1 \\ -6 & 9 & -3 \\ 1 & -3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix}$

$$= -\frac{1}{9}\begin{pmatrix} 36 \\ -72 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -1 \end{pmatrix}.$$

For a single vector this is more work, but if we have several it becomes quite efficient.

Exercise 3: Let α be the standard basis and let β be this new basis. Then $\begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{pmatrix} 4 & -1 & 8 \\ 1 & -2 & -11 \\ 3 & 2 & -7 \end{pmatrix}.$

Hence $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}^{-1} = \left(\frac{1}{234}\right)\begin{pmatrix} 36 & 9 & 27 \\ -26 & -52 & 52 \\ 8 & -11 & -7 \end{pmatrix}.$

$$\text{So} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow \left(\frac{1}{234} \right) \begin{pmatrix} 36 & 9 & 27 \\ -26 & -52 & 52 \\ 8 & -11 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \left(\frac{1}{234} \right) \begin{pmatrix} 135 \\ 26 \\ -35 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} \rightarrow \left(\frac{1}{234} \right) \begin{pmatrix} 36 & 9 & 27 \\ -26 & -52 & 52 \\ 8 & -11 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

$$= \left(\frac{1}{234} \right) \begin{pmatrix} 378 \\ -52 \\ -20 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 6 \\ 10 \end{pmatrix} \rightarrow \left(\frac{1}{234} \right) \begin{pmatrix} 36 & 9 & 27 \\ -26 & -52 & 52 \\ 8 & -11 & -7 \end{pmatrix} \begin{pmatrix} -2 \\ 6 \\ 10 \end{pmatrix}$$

$$= \left(\frac{1}{234} \right) \begin{pmatrix} 252 \\ 260 \\ -152 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 7 \\ 6 \end{pmatrix} \rightarrow \left(\frac{1}{234} \right) \begin{pmatrix} 36 & 9 & 27 \\ -26 & -52 & 52 \\ 8 & -11 & -7 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 6 \end{pmatrix}$$

$$= \left(\frac{1}{234}\right)\begin{pmatrix} 333 \\ -130 \\ -95 \end{pmatrix}.$$

Exercise 4: (i) $\begin{pmatrix} 1 & 2 & -5 \\ 2 & 3 & -1 \\ 0 & 3 & -4 \end{pmatrix}$;

(ii) T(1, 1, 0) = (3, 5, 3), T(2, −1, 2) = (−10, −1, −11), T(3, 0, −4) = (23, 10, 16).

We must now find the coordinates of these images relative to the basis β.

Let (3, 5, 3) = x(1, 1, 0) + y(2, −1, 2) + z(3, 0, −4).

$$\begin{pmatrix} 1 & 2 & 3 & | & 3 \\ 1 & -1 & 0 & | & 5 \\ 0 & 2 & -4 & | & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 3 \\ 0 & -3 & -3 & | & 2 \\ 0 & 2 & -4 & | & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 1 & -11 & | & 8 \\ 0 & 2 & -4 & | & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 1 & -11 & | & 8 \\ 0 & 0 & 18 & | & -13 \end{pmatrix}.$$

So $z = -\dfrac{13}{18}$, $y = 8 - \dfrac{143}{18} = \dfrac{1}{18}$ and $x = 3 - \dfrac{2}{18} + \dfrac{39}{18} = \dfrac{91}{18}$.

Let (−10, −1, −11) = x(1, 1, 0) + y(2, −1, 2) + z(3, 0, −4).

$$\begin{pmatrix} 1 & 2 & 3 & | & -10 \\ 1 & -1 & 0 & | & -1 \\ 0 & 2 & -4 & | & -11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & -10 \\ 0 & -3 & -3 & | & 9 \\ 0 & 2 & -4 & | & -11 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & | & -10 \\ 0 & 1 & -11 & | & -13 \\ 0 & 2 & -4 & | & -11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & -10 \\ 0 & 1 & -11 & | & -13 \\ 0 & 0 & 18 & | & 15 \end{pmatrix}.$$

So $z = \frac{15}{18} = \frac{5}{6}$, $y = -13 + \frac{55}{6} = -\frac{23}{6}$ and

$x = -10 + \frac{46}{6} - \frac{15}{6} = -\frac{29}{6}$.

Let $(23, 10, 16) = x(1, 1, 0) + y(2, -1, 2) + z(3, 0, -4)$.

$$\begin{pmatrix} 1 & 2 & 3 & | & 23 \\ 1 & -1 & 0 & | & 10 \\ 0 & 2 & -4 & | & 16 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 23 \\ 0 & -3 & -3 & | & -13 \\ 0 & 2 & -4 & | & 16 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 23 \\ 0 & 1 & -11 & | & 19 \\ 0 & 2 & -4 & | & 16 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 23 \\ 0 & 1 & -11 & | & 19 \\ 0 & 0 & 18 & | & -22 \end{pmatrix}.$$

So $z = -\frac{11}{9}$, $y = 19 - \frac{121}{9} = \frac{50}{9}$ and

$x = 23 - \frac{100}{9} + \frac{33}{9} = \frac{140}{9}$.

Hence the coordinates of the images with respect to β are, respectively,

$$\frac{1}{18}\begin{pmatrix} 91 \\ 1 \\ -13 \end{pmatrix}, \ \frac{1}{18}\begin{pmatrix} -87 \\ -69 \\ 15 \end{pmatrix}, \ \frac{1}{18}\begin{pmatrix} 280 \\ 100 \\ -22 \end{pmatrix}.$$

The matrix of T with respect to β is therefore

$$\frac{1}{18}\begin{pmatrix} 91 & -87 & 280 \\ 1 & -69 & 100 \\ -13 & 15 & -22 \end{pmatrix}.$$

However it is much easier to use the change of basis theorems.

$$\left[\frac{T(\alpha)}{\alpha}\right] = \begin{pmatrix} 1 & 2 & -5 \\ 2 & 3 & -1 \\ 0 & 3 & -4 \end{pmatrix}$$

$$\left[\frac{\beta}{\alpha}\right] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 2 & -4 \end{pmatrix}.$$

Hence $\left[\dfrac{\beta}{\alpha}\right]^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 2 & -4 \end{pmatrix}^{-1} = \dfrac{1}{18}\begin{pmatrix} 4 & 14 & 3 \\ 4 & -4 & 3 \\ 2 & -2 & -3 \end{pmatrix}.$

$$\left[\frac{T(\beta)}{\alpha}\right] = \left[\frac{\beta}{\alpha}\right]^{-1}\left[\frac{T(\alpha)}{\alpha}\right]\left[\frac{\beta}{\alpha}\right]$$

$$= \frac{1}{18} \begin{pmatrix} 4 & 14 & 3 \\ 4 & -4 & 3 \\ 2 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -5 \\ 2 & 3 & -1 \\ 0 & 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 2 & -4 \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 32 & 59 & -46 \\ -4 & 5 & -28 \\ -2 & -11 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 2 & -4 \end{pmatrix}$$

$$= \frac{1}{18} \begin{pmatrix} 91 & -87 & 280 \\ 1 & -69 & 100 \\ -13 & 15 & -22 \end{pmatrix}.$$

Exercise 5: The kernel of D is the set of all constant functions.

Since $1 \notin U$, $\ker D_U = 0$.

Hence rank$(D_U) = 0$ and so nullity$(D_U) = 2$.

Since $1 \in V$ ker $D_V = \langle 1 \rangle$.

Hence rank$(D_v) = 1$ and so nullity$(D_V) = 1$.

Exercise 6:

(i) $\cos 2x = \cos^2 x - \sin^2 x$ and $x = \frac{1}{2}(2x)$ so

$V = \langle \sin 2x, \sin^2 x, \cos^2 x, x \rangle$. That is, β spans V.

We must now show that β is linearly independent.

Suppose $p \sin 2x + q \sin^2 x + r \cos^2 x + sx = 0$ for all x.

Put $x = 0$: Then $r + s = 0$.   ... (1)

Put x = $\pi/4$: Then $p + \frac{1}{2} q + \frac{1}{2} r + \frac{1}{4} s\pi = 0$ ... (2)

Put $x = \pi/2$: Then $q + \frac{1}{2} s\pi = 0$.   ... (3)

Put $x = \pi$: Then $r + s\pi = 0$.   ... (4)

From (1), (4) we conclude that s = 0. From (1) we now conclude that $r = 0$.

From (3) we conclude that $q = 0$. From (2) we conclude that $p = 0$.

Hence β is a linearly independent set and so is a basis for V.

(ii) We know that differentiation is a linear transformation. What is at issue here is whether its image is a subset of V. Letting D represent differentiation we have:

$$D \sin 2x = 2\cos x.2x = 2\cos^2 x - 2\sin^2 x \in V,$$
$$D \sin^2 x = 2\sin x.\cos x = \sin 2x \in V,$$
$$D \cos^2 x = - 2\sin x.\cos x = - \sin 2x \in V.$$
$$D x = 1 = \sin^2 x + \cos^2 x \in V.$$

(iii) We can now write down the matrix of D relative to the basis β as

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$